

# ON THE TANGENTIAL TOUCH BETWEEN THE FREE AND THE FIXED BOUNDARIES FOR THE TWO-PHASE OBSTACLE-LIKE PROBLEM

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**ABSTRACT.** In this paper we consider the following two-phase obstacle-problem-like equation in the unit half-ball

$$\Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}}, \quad \lambda_{\pm} > 0.$$

We prove that the free boundary touches the fixed one in (uniformly) tangential fashion if the boundary data  $f$  and its first and second derivatives vanish at the touch-point.

## 1. INTRODUCTION

**1.1. The Problem.** The following two-phase analogue of classical obstacle problem was suggested by G. S. Weiss in [W2] and then considered by N.N. Uraltseva in [U] and H. Shahgholian, N.N. Uraltseva and G.S. Weiss in [SUW]. Study properties of a weak solution  $u \in W^{1,2}(D)$  of

$$(1) \quad \Delta u = \lambda_+ \chi_{\{u>0\}} - \lambda_- \chi_{\{u<0\}},$$

in the domain  $D$ , such that  $u - f \in W_0^{1,2}(D)$  for a given  $f \in W^{1,2}(D)$ . In our paper we always assume  $\lambda_{\pm} > 0$ , and we consider the cases  $D = B_1$  and  $D = B_1^+$ , as well as the case of the so-called global solutions  $D = \mathbb{R}_+^n$ .

Obviously (1) is the Euler-Lagrange equation of the energy functional

$$J(u) = \int_D |\nabla u|^2 + 2\lambda_+ \max(u, 0) + 2\lambda_- \max(-u, 0) dx.$$

Note that if the boundary data  $f$  is non-negative (non-positive) then the solution  $u$  is so, too, and we arrive at classical obstacle problem (see [C]). In the two-phase case we do not have the property that the gradient vanishes on the free boundary  $\Gamma_u$  (see Section 1.2 for definition), as it was in the classical case; this causes difficulties.

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*Key words and phrases.* Free boundary problems, two-phase obstacle problem, contact points.

2000 *Mathematics Subject Classification.* Primary 35R35.

<sup>†</sup> Supported by Deutsche Forschungsgemeinschaft and Freistaat Sachsen.

Second and third authors thank Göran Gustafsson Foundation for visiting appointments to KTH, Stockholm.

We consider the following problem: Let  $u$  be a weak solution of (1) in  $B_1^+$ ,  $0 \in \bar{\Gamma}_u$ ,  $f := u|_{\Pi} \in C^{2,Dini}(B_1 \cap \Pi)$  and

$$(2) \quad f(0) = |\nabla f(0)| = |D^2 f(0)| = 0.$$

Then we prove that the free boundary of  $u$  approaches the fixed one at 0 tangentially. Under some growth assumptions we prove that this approach is uniform (Theorem B) and we show the necessity of this assumption with an example. From (2) it obviously follows that  $\frac{|f(x')|}{|x'|^2} \leq \omega(|x'|)$  for some Dini modulus of continuity  $\omega$ , i.e., the blow-up of  $f$  is zero

$$f_r(x') := \frac{f(rx')}{r^2} \rightarrow 0 \text{ as } r \rightarrow 0.$$

Let us recall the definition of  $C^{2,Dini}(B_1 \cap \Pi)$ ; these are functions from  $C^2(B_1 \cap \Pi)$  such that

$$|D^2 f(x) - D^2 f(y)| \leq \omega(|x - y|),$$

where  $D^2 f$  is the Hessian of  $f$  and  $\omega$  is a Dini modulus of continuity, i.e.,

$$\int_0^1 \frac{\omega(s)}{s} ds < \infty.$$

**1.2. Notations.** In the sequel we use following notations:

$\mathbf{R}_+^n$	$\{x \in \mathbf{R}^n : x_1 > 0\}$
$\mathbf{R}_-^n$	$\{x \in \mathbf{R}^n : x_1 < 0\},$
$B(z, r)$	$\{x \in \mathbf{R}^n :  x - z  < r\},$
$B_r$	$B(0, r),$
$B_r^+$	$\mathbf{R}_+^n \cap B_r,$
$\Pi$	$\{x \in \mathbf{R}^n : x_1 = 0\},$
$x'$	$(x_2, \dots, x_n),$
$K_\epsilon$	$\{x \in \mathbf{R}_+^n : x_1 > \epsilon x' \},$
$\ \cdot\ _\infty$	canonical norm,
$e_1, \dots, e_n$	standard basis in $\mathbf{R}^n$ ,
$\nu, e$	arbitrary unit vectors,
$D_\nu, D_{\nu e}$	first and second directional derivatives,
$v^+, v^-$	$\max(v, 0), \max(-v, 0),$
$\chi_D$	characteristic function of the set $D$ ,
$\partial D$	boundary of the set $D$ ,
$\Omega_u^+$	$\{x \in D : u(x) > 0\},$
$\Omega_u^-$	$\{x \in D : u(x) < 0\},$
$\Lambda_u$	$\{x \in B_1^+ : u(x) =  \nabla u(x)  = 0\},$
$\Gamma_u$	$(\partial\Omega_u^+ \cup \partial\Omega_u^-) \cap D$ , the free boundary,
$\mathcal{P}(\dots)$	see Definition 2.

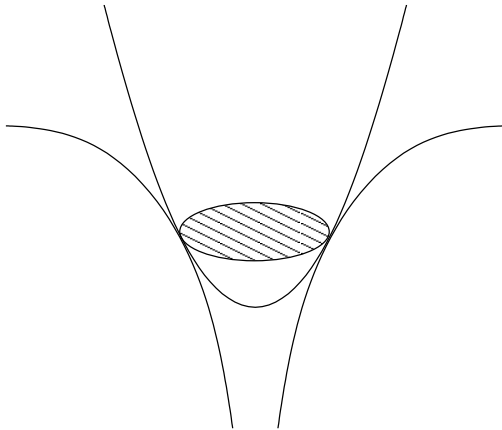


FIGURE 1. “Ball solutions”

**1.3. “Typical” examples.** We show here with some examples how the situation near a touch point between free and the fixed boundaries can look like.

Fix the ball  $B_R$  and consider the function  $\frac{\lambda}{2n}|x|^2$ ,  $\lambda > 0$ . Let us take the radial fundamental solution of Laplace equation  $U$  multiplied with a constant  $C_R$ , such that  $C_R \partial_r U(R) = \frac{\lambda}{n}R$ . Then for some constant  $C$  the function

$$V(x) = \frac{\lambda}{2n}|x|^2 - C_R U(|x|) + C$$

is non-negative in  $\mathbb{R}^n$ ,  $\Delta V = \lambda - C_R \delta_0$  and  $V = |\nabla V| = 0$  on  $\partial B_R$  (Figure 1).

Now we see how we can construct solutions of (1) in  $B_R \setminus \{0\}$  and in  $\mathbb{R}^n \setminus B_R$ . So for instance in  $\mathbb{R}^2$  we can illustrate some solutions considered in rectangles (Figure 2). The dashed curves denote free boundaries,  $\pm$  denote regions  $\Omega_u^\pm$  and 0 the region  $\Lambda_u$ . Figure 2 a) shows that the case when the solution does not have quadratic growth near the touch point is possible. In this case the blow-up of the solution is zero. Figure 2 b) shows that even if we have non-negative boundary data near touch point, the blow-up still can be negative and Figure 2 c) shows that the condition (2) is essential for the tangential touch.

Let us take boundary data  $f$  on  $\partial B_1^+$  to be odd-symmetric with respect to  $x_2$ . Then the solution  $u$  will be odd-symmetric, too. From our results (see Section 2) follows that the set  $\Lambda_u$  is large near 0, where the free boundary touches the fixed boundary, as it is the case in the example from the Figure 2 a). So we do not have orthogonal touch, as it might be expected. The similar argument works also in higher dimensions for every plane-symmetric domain.

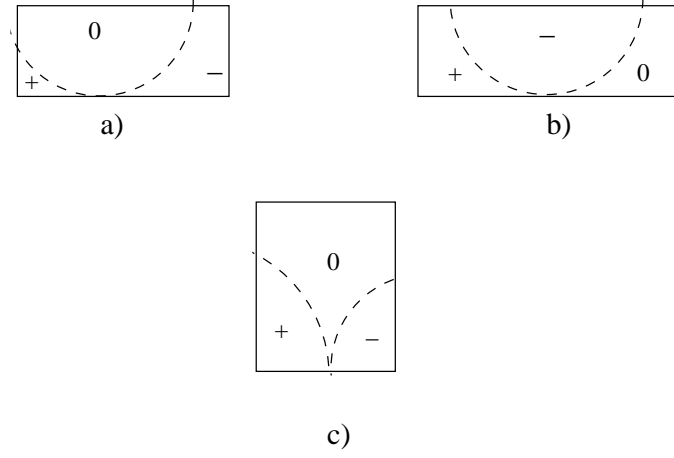


FIGURE 2. “Typical” examples

## 2. MAIN RESULTS

In this section we state two theorems. The first one says that if the boundary data of a solution of (1) satisfies condition (2), then the free boundary can approach the fixed one only tangentially. In the second theorem we assert that this approach is uniform for a certain class of functions.

**Theorem A.** *Let  $u$  be a solution of (1) in  $B_1^+$  with boundary data  $f$  on  $\Pi$ , condition (2) is satisfied and  $0 \in \overline{\Gamma}_u$ . Then the free boundary approaches  $\Pi$  at the point 0 tangentially.*

**Corollary 1.** *Let  $u$  be as in Theorem A, then one of the following expressions is true*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \rightarrow 1, \quad \frac{|\Omega_u^- \cap B_r^+|}{|B_r^+|} \rightarrow 1, \quad \frac{|\Lambda_u \cap B_r^+|}{|B_r^+|} \rightarrow 1 \text{ as } r \rightarrow 0.$$

Moreover, the first two cases are possible only if the condition (4), see below, is satisfied for some  $c_0$  and  $r_0$ , and the third case if it fails to hold (see Lemma 8).

**Definition 2.** *Let  $\omega$  be a Dini modulus of continuity and  $M, c_0, r_0$  be positive constants. We define  $\mathcal{P}(M, R, c_0, r_0)$  to be the class of solutions  $u$  of (1) in  $B_1^+$ ,  $\|u\|_{L^\infty(B_1^+)} \leq M$ ,  $0 \in \overline{\Gamma}_u$  such that the boundary data*

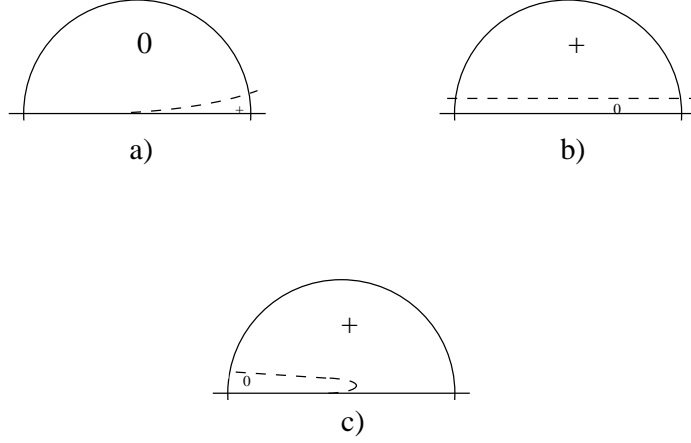


FIGURE 3. Non-uniform approach

$f = u|_{\Pi} \in C^{2,Dini}(B_1 \cap \Pi)$  satisfies condition (2) and

$$(3) \quad \|f\|_{C^2(\overline{B}_1 \cap \Pi)} \leq R, \quad \int_0^1 \frac{\omega(s)}{s} ds \leq R.$$

We assume, further

$$(4) \quad \sup_{B_r^+} |u| \geq c_0 r^2, \text{ for } 0 < r < r_0,$$

for all  $u \in \mathcal{P}(M, R, c_0, r_0)$ .

**Remark 3.** If  $u$  solve (1) in  $B_1^+$ ,  $0 \in \overline{\Gamma}_u$  and  $u|_{\Pi} \equiv 0$ , then condition (4) is fulfilled with the constant  $c_0 = \max(\lambda_{\pm})C$ , for  $0 < r < 1$ , where  $C$  is a dimension dependent constant (see Lemma 6 and Corollary 7).

**Theorem B.** There exists a modulus of continuity  $\sigma(r)$  and  $\tilde{r} > 0$  such that if  $u \in \mathcal{P}(M, R, c_0, r_0)$  then

$$\Gamma_u \cap B_{\tilde{r}} \subset \{x : x_1 < |x'| \sigma(|x'|)\}.$$

In other words the free boundary of the functions from  $\mathcal{P}$  approaches  $\Pi$  at the point 0 uniformly tangentially.

Here  $\sigma$  and  $\tilde{r}$  depend on the dimension,  $\lambda_{\pm}$ ,  $c_0$ ,  $r_0$ ,  $M$  and  $R$ .

**Remark 4.** Since the main tool we use proving Theorems A and B is the blow-up argument, these results could be generalized for domains with smooth enough boundary.

The following example shows that we do not have uniform tangential approach when the set  $\Lambda_u$  is large near the touch point, i.e., the condition (4) fails to hold (see Lemma 8). In the example the boundary data is positive, so we treat here the classical obstacle problem in  $\mathbb{R}^2$ . First let us take small positive boundary data on the right side from touch point (zero) to get a tiny positivity set as it is shown on the Figure 3 a). This can be done using the “ball solutions” above. Next consider the function  $u(x) = \frac{\lambda_+}{2}((x_1 - \epsilon)_+)^2$ . We will get it as a solution if we take its boundary data on  $\partial B_1^+$  (Figure 3 b)). Consider now the boundary data which is the sum of the boundary data of the previous two examples. It will look like it is shown on the Figure 3 c). So we see that when the ‘tiny’ positivity set in Figure 3 a) becomes smaller and  $\epsilon$  in Figure 3 b) tends to zero, then for any  $\tilde{r}$  and  $\sigma$  in the Figure 3 c) we get free boundary points in  $B_{\tilde{r}} \cap \{x : x_1 > |x'| \sigma(|x'|)\}$ , while the boundary data on  $\Pi$  remains bounded.

**Remark 5.** *In order to get uniform tangential touch for a class of solutions we impose condition (4). This condition, however, can be replaced by the following one, which is considered in [KKS] for a different problem;*

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \geq c_0 > 0, \text{ for } r < r_0.$$

*From Lemma 8 and Corollary 1 it follows that both conditions are equivalent in our case.*

### 3. TECHNICALITIES

**3.1. Non-degeneracy.** In this section we introduce some (modified) results from [W2], [U] and [SUW] as well as prove growth estimates at the boundary (Lemmas 8 and 9).

**Lemma 6.** *Let  $u$  solve (1) in  $B_1$ . There exists a dimension dependent constant  $C$  such that  $\|f^\pm\|_\infty < \lambda_\pm C$  implies  $0 \notin \overline{\Omega}_u^\pm$ .*

**Proof.** Consider the “+”-case. Due to the comparison principle a similar argument is true (and well-known) for obstacle problems, i.e. it is known in our case if the boundary data  $f$  is non-negative or non-positive. Let us now consider the related (one-phase) obstacle problem in  $B_1$  with boundary data  $f^+$ , denote its solution by  $v$ . It is enough to show that  $\Omega_u^+ \subset \Omega_v^+$ . Consider the function  $w = u - v$  in  $\Omega_u^+$ . We have  $w|_{\partial\Omega_u^+} \leq 0$  and  $\Delta w \geq 0$  in  $\Omega_u^+$ , hence  $u - v \leq 0$  and we are done.  $\square$

**Corollary 7.** *Let  $u$  be the solution of (1),  $x_0 \in \overline{\Omega}^\pm$  and  $B_{r_0}(x_0) \subset D$ . Then*

$$(5) \quad \sup_{\partial B_r(x_0)} u^\pm \geq \lambda_\pm C r^2, \text{ for } r < r_0.$$

Here the constant  $C$  is the same as in the previous lemma.  
In other words if  $x_0 \notin \text{int } \Lambda_u$  and  $B_{r_0}(x_0) \subset D$ , then

$$(6) \quad \sup_{\partial B_r(x_0)} |u| \geq \min(\lambda_{\pm}) Cr^2, \text{ for } r < r_0.$$

**Proof.** Let us restrict the function  $u$  to  $B_r(x_0)$  and scale

$$u_r(x) = \frac{u(rx + x_0)}{r^2}.$$

$u_r$  is then a solution of (1) in  $B_1$  with boundary data  $u_r|_{\partial B_1}$ . Since  $0 \in \bar{\Omega}^+$  we must have

$$\sup_{\partial B_1} u_r \geq \lambda_+ C,$$

which in turn implies (5).  $\square$

**Lemma 8.** Let  $u$  be the solution of (1) in  $B_1^+$  and suppose for given constants  $c_0, r_0$ , we have

$$\frac{|\Omega_u^+ \cap B_r^+|}{|B_r^+|} \geq c_0 > 0, \text{ for } r < r_0.$$

Then there exists a constant  $c$  depending on,  $c_0, \lambda_{\pm}$  and the dimension such that

$$(7) \quad \sup_{B_r^+} |u| \geq cr^2, \text{ for } r < r_0.$$

The same is also true for  $\Omega^-$ .

**Proof.** Let us denote  $\tilde{B}_r^+ = B_r^+ \cap \{x_1 > \epsilon r\}$ . We can fix an  $\epsilon > 0$  such that

$$\frac{|\Omega_u^+ \cap \tilde{B}_r^+|}{|\tilde{B}_r^+|} \geq \frac{c_0}{2}, \text{ for } r < r_0.$$

Hence for each  $r > 0$  there exists  $x_r \in \Omega_u^+ \cap \tilde{B}_{\frac{r}{2}}^+$ . Applying previous corollary to the ball  $B_{d_r}(x_r)$ , where  $d_r$  is the  $e_1$  component of  $x_r$  we get that

$$\sup_{B_r^+} |u| \geq \sup_{B_{d_r}(x_r)} |u| \geq \lambda_+ C d_r^2 \geq \epsilon \lambda_+ \frac{C}{4} r^2.$$

We proved the lemma with  $c = \epsilon \lambda_+ \frac{C}{4}$ .  $\square$

In the proof of the next lemma we use the technique from [A] (Lemma 5), see also [CKS]. A similar estimate in the interior was proved by Uraltseva in [U].

**Lemma 9.** Let  $u$  solve (1) in  $B_1^+$ ,  $\|u\| \leq M$  and assume its boundary data  $f = u|_{\Pi}$  and the Dini modulus of continuity  $\omega$  satisfy conditions (2) and (3). Then there exists a constant  $C = C(M, R)$  such that

$$\sup_{B_r^+} |u - D_{e_1} u(0) x_1| \leq Cr^2, \quad 0 < r < \frac{1}{2}.$$

**Proof.** Let us denote by

$$S_j(u) := \sup_{B_{2^{-j}}^+} |u - D_{e_1} u(0)x_1|$$

and  $\mathbb{M}(u) := \{j : S_j(u) \leq 4S_{j+1}(u)\}$ . We want to show that  $S_j(u) \leq C2^{-2j}$ . First let us show this for all  $j \in \mathbb{M}(u)$ . The proof is done by contradictory argument: assume there exist a sequence  $\{u_j\}$  of solutions of (1) in  $B_1^+$  such that

$$S_{k_j}(u_j) \geq j2^{-2k_j},$$

for some  $k_j \in \mathbb{M}(u_j)$ . Denoting by  $w_j(x) := u_j(x) - D_{e_1} u_j(0)x_1$  and by

$$\tilde{w}_j(x) := \frac{w_j(2^{-k_j}x)}{S_{k_j+1}(u_j)},$$

we get

$$\|\Delta \tilde{w}_j\|_\infty \leq \max(\lambda_\pm) \frac{2^{-2k_j}}{S_{k_j+1}(u_j)} \leq \max(\lambda_\pm) \frac{2^{-2k_j}}{\frac{1}{4}S_{k_j}(u_j)} \leq \max(\lambda_\pm) \frac{4}{j} \rightarrow 0.$$

We also have

$$(8) \quad \sup_{B_{\frac{1}{2}}^+} |\tilde{w}_j| = 1.$$

The condition  $(D_e f(x'))^\pm \leq |x'| \omega(|x'|)$ , for any unit vector  $e \in \Pi$ , implies that

$$(9) \quad \sup_{B_r^+} |D_e w_j| \leq Cr,$$

where  $C$  depends on  $M$  and  $R$ . To see this we should consider harmonic functions  $v_j^\pm$  in  $B_{\frac{1}{2}}^+$  with the same boundary data as  $(D_e w_j)^\pm$ . Inequality (9) then follows from the subharmonicity of  $(D_e w_j)^\pm$  (see [U]) and standard estimates on Green's function for the half-ball (see [Wi]). From (9) we have

$$(10) \quad \sup_{B_r^+} |D_e \tilde{w}_j| \leq \frac{4Cr}{j},$$

A subsequence of  $\tilde{w}_j$  converges in  $C^1(B_{\frac{1}{2}}^+)$  to a harmonic function  $u_0$ . Due to (10) we get  $D_e u_0 = 0$  for all  $e \in \Pi$ , thus  $u_0 = ax_1$ . On the other hand  $D_{e_1} \tilde{w}_j(0) = 0$  and by  $C^1$ -convergence (up to  $\Pi$ ) the same holds for  $u_0$ . Hence  $u_0 \equiv 0$  which contradicts (8).

Next let us show that  $S_j(u) \leq 4C2^{-2j}$  for all  $j$ . Suppose  $j$  is the first integer for which the inequality fails to hold, then

$$S_{j-1}(u) \leq 4C2^{-2(j-1)} \leq 4S_j(u),$$

i.e.  $j-1 \in \mathbb{M}(u)$  and

$$S_j(u) \leq S_{j-1}(u) \leq C2^{-2(j-1)} = 4C2^{-2j},$$

a contradiction. □



**3.2. Monotonicity formulae.** Here we introduce two monotonicity formulae in the following two lemmas, which play crucial role in our proofs. The first one was presented by H. W. Alt, L. A. Caffarelli and A. Friedman in [ACF] and was developed then in [CKS]. The second one is due to G. Weiss [W1], [SUW]. In [A] Andersson adapted it to the half-space case and our representation is analogous. See also [M] for the formula in parabolic case.

**Lemma 10. (ACF monotonicity formula)**

Let  $h_1, h_2$  be two non-negative continuous subsolutions of  $\Delta u = 0$  in  $B_R$ . Assume further that  $h_1 h_2 = 0$  and that  $h_1(0) = h_2(0) = 0$ . Then the following function is non-decreasing in  $r \in (0, R)$

$$(11) \quad \varphi(r) = \frac{1}{r^4} \left( \int_{B_r} \frac{|\nabla h_1|^2 dx}{|x|^{n-2}} \right) \left( \int_{B_r} \frac{|\nabla h_2|^2 dx}{|x|^{n-2}} \right).$$

More exactly, if any of the sets  $\text{spt}(h_j) \cap \partial B_r$  digresses from a spherical cap by a positive area, then either  $\varphi'(r) > 0$  or  $\varphi(r) = 0$ .

**Lemma 11. (Weiss' monotonicity formula)**

Let  $u$  solve (1) in  $B_R^+$  and  $u|_{\Pi \cap B_R} = 0$ . Then the function

$$(12) \quad \Phi(r) = r^{-n-2} \int_{B_r \cap \mathbb{R}_+^n} (|\nabla u|^2 + 2\lambda_+ u^+ + 2\lambda_- u^-) - r^{-n-3} \int_{\partial B_r \cap \mathbb{R}_+^n} 2u^2 d\mathcal{H}^{n-1}$$

is non-decreasing for  $r \in (0, R)$ . Moreover, if  $\Phi(\rho) = \Phi(\sigma)$  for any  $0 < \rho < \sigma < R$  then  $\Phi$  is homogeneous of degree two in  $(B_\sigma \setminus B_\rho) \cap \mathbb{R}_+^n$ .

The proof is analogous to the proof of the Lemma 1 in [A].

#### 4. GLOBAL SOLUTIONS

In this section we will classify all solutions of (1) in the  $\mathbb{R}_+^n$  with zero boundary data and quadratic growth. We will see that only possible cases are

$$(13) \quad u(x) = \pm \frac{\lambda_\pm}{2} ((x_1 - a)_+)^2, \quad a \geq 0 \quad \text{or} \quad u(x) = \pm \frac{\lambda_\pm}{2} x_1^2 \pm \alpha x_1, \quad \alpha \geq 0.$$

The proofs of next two lemmas adapt the proofs of analogous results from [SU] for our case.

Let us first prove that  $u$  is two-dimensional.

**Lemma 12.** Let  $u$  solve (1) in  $\mathbb{R}_+^n$  with boundary data  $u|_{\Pi} = 0$ . Then the function  $u$  is two-dimensional, i.e., in some system of coordinates

$$u(x) = u(x_1, x_2),$$

where the  $e_1$  direction is normal to  $\Pi$ .

**Proof.** Let us take any  $e$  orthogonal to  $e_1$  and consider functions  $(D_e u)^\pm$ . In [U] Uraltseva proved that these functions are subharmonic. Note that they will remain so if we extend them by zero to  $\mathbb{R}_+^n$ . Now we can apply ACF monotonicity formula to  $(D_e u)^\pm$ . For  $r < s$  we have

$$\varphi(r, D_e u) \leq \varphi(s, D_e u) \leq \lim_{s \rightarrow \infty} \varphi(s, D_e u) =: C_e.$$

In [U] is shown that the second derivatives of  $u$  are bounded, thus we can find a sequence  $u_{r_j} = \frac{u(r_j x)}{r_j^2} \rightarrow u_\infty$ , uniformly on compact subsets and in  $(W_{loc}^{2,p} \cap C_{loc}^{1,\alpha})(\mathbb{R}_+^n \cup \Pi)$ , for any  $1 < p < \infty$  and  $0 < \alpha < 1$ . We have now

$$C_e = \lim_{r_j \rightarrow \infty} \varphi(sr_j, D_e u) = \lim_{r_j \rightarrow \infty} \varphi(s, D_e u_{r_j}) = \varphi(s, D_e u_\infty), \quad \forall s > 0.$$

From  $\{x_1 < 0\} \subset \{D_e u = 0\}$  follows that  $\varphi(r, D_e u_\infty) \equiv 0$  or  $\varphi'(r, D_e u_\infty) > 0$  for all  $r > 0$ , thus  $C_e = 0$  and we get  $D_e u \geq 0$  or  $D_e u \leq 0$ .

For  $e_2 \in \Pi$  assume  $D_{e_2} u \geq 0$  and let  $e_3 \in \Pi$  be orthogonal to  $e_2$ . Consider unit vector  $e(\phi) = \cos \phi e_2 + \sin \phi e_3 \in \Pi$ ,  $\phi \in [0, \pi]$ . From the  $C^1$ -continuity we have that the sets  $\{\phi : \Omega_{D_{e(\phi)} u}^\pm \neq \emptyset\}$  are relatively open in  $[0, \pi]$ . On the other hand they are both non-empty and have empty intersection; this means that there exists  $\phi_0 \in (0, \pi)$  such that  $D_{e(\phi_0)} u \equiv 0$ . Rotating coordinates we can get  $D_{e_2} u \geq 0$  and  $D_{e_3} u \equiv 0$ . Repeating this with  $e_k$ ,  $k = 4, \dots, n$ , we get that  $u$  is two dimensional.  $\square$

We prove now the main result of this section under the assumption of homogeneity.

**Proposition 13.** *Let  $u$  be homogeneous of degree two and solve (1) in  $\mathbb{R}_+^n$  with boundary data  $u|_\Pi = 0$ . Then either  $u(x) = \frac{\lambda_+}{2} x_1^2$  or  $u(x) = -\frac{\lambda_-}{2} x_1^2$ .*

**Proof.** We can consider only two-dimensional functions  $u$ . So let us rewrite  $u$  in radial coordinates as

$$u(x) = u(r, \theta) = r^2 \phi(\theta), \quad r \in [0, \infty), \quad \theta \in [0, \pi].$$

Then we get the ODE

$$\phi'' + 4\phi = \lambda_+ \chi_{\{\phi > 0\}} - \lambda_- \chi_{\{\phi < 0\}}$$

in the interval  $[0, \pi]$  with boundary data  $\phi(0) = \phi(\pi) = 0$ . It can be checked that the only solutions of this ODE are  $\phi(\theta) = \pm \frac{\lambda_\pm}{2} \sin^2 \theta$ .  $\square$

**Lemma 14.** *Let  $u$  solve (1) in  $\mathbb{R}_+^n$  with boundary data  $u|_\Pi = 0$  and be quadratically bounded at infinity. Then  $u$  is one of the representations in (13).*

**Proof.** If the function  $u$  is non-negative or non-positive, then the result we want to prove follows from Theorem B in [SU]. So let us show that

$u$  does not change the sign. We do this by contradiction; assume that  $u^\pm$  are both non-trivial.

Consider the shrink down of  $u$ ;  $\tilde{u} := \lim_{j \rightarrow \infty} u_j$ , where  $u_j(x) = \frac{u(r_j x)}{r_j^2}$ ,  $r_j \rightarrow \infty$ . It is homogeneous of degree two. To see this we need to use Weiss' monotonicity formula

$$\Phi(s, \tilde{u}) = \lim_{j \rightarrow \infty} \Phi(s, u_j) = \lim_{j \rightarrow \infty} \Phi(sr_j, u) = \Phi(\infty, u).$$

Thus  $\tilde{u}$  equals to one of  $\pm \frac{\lambda_\pm}{2} x_1^2$  by Proposition 13 above. Assume for definiteness that we have the “+”-sign.

This means that for any  $\delta > 0$  there exists  $R_\delta$  such that

$$(14) \quad \Omega_u^- \setminus B_{R_\delta}^+ \subset \{x; x_1 < \delta|x_2|\}.$$

Let us now take the barrier function

$$U(x_1, x_2) = x_1^4 + x_2^4 - 6x_1^2 x_2^2 + C.$$

For large enough  $C$  we have  $\Omega_u^- \Subset \Omega_U^+$ . Since  $u$  is quadratically bounded we get from the comparison principle that  $u^-(x) \leq \varepsilon U(x)$  for any  $\varepsilon > 0$ , thus  $\Omega_u^- = \emptyset$ .  $\square$

## 5. PROOFS

**Proof of the Theorem A.** We consider here only the case when (4) fails to hold. From Lemma 9 follows that  $D_{x_1} u(0) = 0$  and

$$(15) \quad \sup_{B_r^+} |u| \leq c_0 r^2, \text{ for } r < r_0.$$

Now assume that we do not have tangential touch at 0, i.e., there is an  $\epsilon > 0$  and a sequence  $x^j \in K_\epsilon \cap \Gamma_u$ ,  $x^j \rightarrow 0$ . Repeating the proof of Lemma 8 we obtain

$$(16) \quad \sup_{B_{2d_j}^+} |u| \geq C d_j^2, \text{ for } r < r_1,$$

where  $d_j = |x^j|$ . Consider now the blow up sequence

$$\tilde{u}_j(x) = \frac{u(2d_j x)}{4d_j^2},$$

which is bounded by (15). Therefore there is a subsequence converging in  $C^{1,\alpha}$  to a global solution  $u_0$  with zero boundary data, which is non-trivial (due to (16)). As in the proof of Lemma 14, using Weiss' monotonicity formula we get that  $u_0$  is homogeneous of degree 2, this implies that  $u_0(x) = \pm \frac{\lambda_\pm}{2} x_1^2$  which contradicts the fact that  $x^j \in K_\epsilon$ .  $\square$

**Proof of the Theorem B.** The proof is done by contradictory argument. Assume there exist an  $\epsilon > 0$ , functions  $u_j$  satisfying the conditions of the theorem and a sequence  $x^j \rightarrow 0$  such that  $x^j \in K_\epsilon \cap \Gamma_{u_j}$ . Let us consider the blow-up sequence

$$\tilde{u}_j(x) = \frac{u_j(d_j x)}{\sup_{B_{d_j}^+} |u_j|},$$

where  $d_j := |x^j|$ . We have that

$$\Delta \tilde{u}_j = \frac{d_j^2}{\sup_{B_{d_j}^+} |u_j|} \Delta u_j.$$

Two cases are possible: either

$$(17) \quad \frac{d_j^2}{\sup_{B_{d_j}^+} |u_j|} \rightarrow 0$$

for some subsequence or

$$(18) \quad \frac{d_j^2}{\sup_{B_{d_j}^+} |u_j|} \not\rightarrow 0$$

for all subsequences.

Let us consider the first case. From Lemma 9 it follows that

$$(19) \quad -Cr^2 + |D_{e_1} u_j(0)|r \leq \sup_{B_r^+} |u_j| \leq Cr^2 + |D_{e_1} u_j(0)|r.$$

This together with (17) gives that  $|D_{e_1} u_j(0)|d_j^{-1} \rightarrow \infty$ , so we can assume

$$(20) \quad |D_{e_1} u_j(0)| > jd_j.$$

From here, and (19) we obtain

$$\left| \frac{\sup_{B_{d_j}^+} |u_j|}{d_j |D_{e_1} u_j(0)|} - 1 \right| \leq \frac{C}{j} \rightarrow 0.$$

We arrive at

$$(21) \quad \sup_{B_r^+} |\tilde{u}_j| = \frac{\sup_{B_{rd_j}^+} |u_j|}{\sup_{B_{d_j}^+} |u_j|} \leq \frac{Cr^2 d_j^2 + |D_{e_1} u_j(0)|rd_j}{\sup_{B_{d_j}^+} |u_j|} \rightarrow r.$$

There is a subsequence of  $\tilde{u}_j$  converging to a function  $u_0$  in  $C^{1,\alpha}$ , that is harmonic in  $\mathbb{R}_+^n$  (due to (17)), linearly bounded (due to (21)) and has zero boundary data at  $\Pi$ . Extending  $u_0$  by odd reflection to  $\mathbb{R}_-^n$  and using Liouville's theorem we get that  $u_0(x) = D_{e_1} u_0(0)x_1$  which contradicts the fact of existence of zeros in  $K_\epsilon$ .

In the case (18) we can without loss of generality assume

$$(22) \quad \frac{d_j^2}{\sup_{B_{d_j}^+} |u_j|} \rightarrow d > 0.$$

We have then that a subsequence of  $\tilde{u}_j$  converges to a function  $u_0$  in  $C^1$  and (22) implies that  $u_0$  is a global solution with  $d\lambda_{\pm}$  instead of  $\lambda_{\pm}$  and zero boundary data. Condition (4) and Lemma 14 give us that  $u_0$  is strictly positive or negative in  $\mathbb{R}_+^n$ , which contradicts the fact that  $x^j \in K_{\epsilon} \cap \Gamma_{u_j}$ . More precisely, functions  $\tilde{u}_j$  vanish at  $\tilde{x}_j := d_j^{-1}x_j \in K_{\epsilon} \cap \Gamma_{u_j} \cap \partial B_1$ , thus we can always choose the subsequence of  $\tilde{u}_j$  in such a way that the corresponding subsequence  $\tilde{x}_j \rightarrow x_0 \in K_{\epsilon} \cap \Gamma_{u_j} \cap \partial B_1$  and then  $u_0(x_0) = 0$ .  $\square$

**Acknowledgment.** The authors are grateful to Prof. H. Shahgholian for valuable discussions and his kind hospitality.

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